Journal of Geometry and Physics 12 (1993) 205–215
North-Holland

JOURNAL OF
GEOMETRY AND
PHYSICS

Appearance of universal bundle structure in four dimensional topological gravity

Siye Wu [1]

Department of Mathematics, Columbia University, New York, NY 10027, USA

Received 10 September 1992
(Revised 8 February 1993)

We calculate the connection and curvature of the universal fibration of the Riemannian manifolds and compare them with the BSRT algebra of four dimensional topological gravity. We also comment on the dimension of the moduli space, conformal gauge fixing and the construction of non-trivial observables.

Keywords: *Topological quantum field theory, fibration of Riemannian manifolds, moduli space of anti-self-dual metrics*
1991 MSC: *81 T 70, 55 R 05, 58 D 17, 58 D 27.*

1. Introduction

Recently, there has been considerable interest in the relation of low dimensional topology and quantum field theory. A typical example is topological Yang–Mills theory [1], where Donaldson's polynomials appear naturally. In the same spirit, four dimensional topological gravity was proposed in refs. [2], followed by various other attempts [3–5]. In this paper, we interpret the BRST algebra as the connection and curvature of a certain universal fiber bundle restricted to the moduli space of anti-self-dual gravity.

The paper is organized as follows: In section 2, we review the BSRT algebra and the classical action. In section 3, we study the universal fibration of Riemannian manifolds and calculate its natural connection and curvature explicitly. The result is compared with the previous section. In section 4, we discuss the moduli space, conformal gauge fixing and observables.

[1] E-mail address: sw@shire.math.columbia.edu

2. BRST transformations and classical action

Witten's action for topological Yang–Mills theory [1] can be written as a topological action (the first Pontryagin class) plus gauge fixing terms [6]. But for gravity, the only local expression with up to second order derivatives of the metric is the Einstein–Hilbert action, which in four dimensions is not a topological invariant. We therefore start with a Lagrangian $\mathcal{L}_0 = 0$ but specify the fields and symmetries (or more precisely, the redundancies). The zero action should be regarded as a functional of the metric $g_{\mu\nu}$ that is invariant under arbitrary local variation in addition to the diffeomorphism and conformal transformations.

We postpone the discussion of conformal gauge fixing to section 4 and start with the diffeomorphism BRST sector [4]:

$$\delta g_{\mu\nu} = \psi_{\mu\nu} + c_{\mu;\nu} + c_{\nu;\mu}, \tag{2.1}$$

$$\delta\psi_{\mu\nu} = \phi_{\mu;\nu} + \phi_{\nu;\mu} - (\psi_{\mu\lambda}c^{\lambda}{}_{;\nu} + \psi_{\nu\lambda}c^{\lambda}{}_{;\mu} + \psi_{\mu\nu;\lambda}c^{\lambda}), \tag{2.2}$$

$$\delta c^{\mu} = -\phi^{\mu} - c^{\mu}{}_{;\nu}c^{\nu}, \tag{2.3}$$

$$\delta\phi^{\mu} = \phi^{\mu}{}_{;\nu}c^{\nu} - c^{\mu}{}_{;\nu}\phi^{\nu}. \tag{2.4}$$

Here $g_{\mu\nu}$ is the Riemannian metric tensor, $\psi_{\mu\nu}$ is a fermionic symmetric tensor, c^{μ} is a fermionic vector field and ϕ^{μ} is a bosonic one. They have ghost numbers 0, 1, 1, 2, respectively. One can check directly that $\delta^2 = 0$. The above transformations can be written in a compact geometric form [4]. Let $\tilde{\mathrm{d}} = \mathrm{d} + \delta$ be the extended "differentiation" and

$$\tilde{\Gamma}^{\mu}_{\nu} = \Gamma^{\mu}_{\lambda\nu}\mathrm{d}x^{\lambda} + (\tfrac{1}{2}\psi^{\mu}{}_{\nu} + c^{\mu}{}_{;\nu}) \tag{2.5}$$

the "connection", then the "curvature" is

$$\begin{aligned}\tilde{R}^{\mu}{}_{\nu} &\equiv \tilde{\mathrm{d}}\tilde{\Gamma}^{\mu}_{\nu} + \tilde{\Gamma}^{\mu}_{\lambda}\wedge\tilde{\Gamma}^{\lambda}_{\nu} \\ &= \tfrac{1}{2}R^{\mu}{}_{\nu\rho\sigma}\,\mathrm{d}x^{\rho}\wedge\mathrm{d}x^{\sigma} + (P^{\mu}{}_{\nu\lambda} - R^{\mu}{}_{\nu\lambda\rho}c^{\rho})\,\mathrm{d}x^{\lambda} \\ &\quad + \tfrac{1}{2}(Q^{\mu}{}_{\nu} + 2P^{\mu}{}_{\nu\rho}c^{\rho} + R^{\mu}{}_{\nu\rho\sigma}c^{\rho}c^{\sigma}),\end{aligned} \tag{2.6}$$

where $R_{\mu\nu\rho\sigma}$ is the standard Riemann curvature of $g_{\mu\nu}$ and

$$P_{\mu\nu\lambda} = \tfrac{1}{2}(\psi_{\lambda\mu;\nu} - \psi_{\lambda\nu;\mu}), \tag{2.7}$$

$$Q_{\mu\nu} = -\tfrac{1}{2}\psi_{\mu\lambda}\psi^{\lambda}{}_{\nu} - (\phi_{\mu;\nu} - \phi_{\nu;\mu}). \tag{2.8}$$

Formulas (2.6)–(2.8) will be interpreted as the curvature of universal fibration in section 3. Unlike topological gauge theory, the components of the "curvature" in (2.6) depend explicitly on the diffeomorphism ghost c^{λ}. If we write

$$\tilde{R}^{\mu}{}_{\nu}\wedge\tilde{R}^{\nu}{}_{\mu} = W_4 + W_3 + W_2 + W_1 + W_0, \tag{2.9}$$

according to the gradings of the ghost number and differential forms, where W_k is a k-form of ghost number $(4-k)$, then $\tilde{\mathrm{d}}(\tilde{R}^{\mu}{}_{\nu}\wedge\tilde{R}^{\nu}{}_{\mu}) = 0$ implies

$$\delta W_k = -\mathrm{d}W_{k-1} \quad (1 \le k \le 4), \qquad \delta W_0 = 0. \tag{2.10}$$

To write the theory in a manageable size, we will ignore the ghosts from diffeomorphism and conformal gauge fixing. Setting $c_\mu = 0$ in (2.1)–(2.4), the operator δ is nilpotent up to a diffeomorphism, i.e., $\delta^2 X = L_\phi X$ for all fields X (including the additional ones defined later). Here L_ϕ is the Lie derivative with respect to the vector field ϕ. We write the gauge fixed Lagrangian as

$$\mathcal{L} = \delta(\sqrt{g}\,\chi^{+\alpha\beta\gamma\delta} W^{+}_{\alpha\beta\gamma\delta}) + \delta(\sqrt{g}\,\overline{\phi}_\lambda (K^* \psi)^\lambda) + \delta(\sqrt{g}\,(\tfrac{1}{2} p \psi^\lambda{}_\lambda + q)), \quad (2.11)$$

where $W^{+}_{\alpha\beta\gamma\delta}$ is the self-dual part of the Weyl tensor and K is the Killing operator

$$K : \mathrm{Vect}(M) \to \mathrm{Sym}^2(T^*M), \quad K : v^\lambda \mapsto v_{\mu;\nu} + v_{\nu;\mu}.$$

The new fields are $\chi^{+}_{\alpha\beta\gamma\delta}$, $b^{+}_{\alpha\beta\gamma\delta}$ (which have the same algebraic properties as W^+); two vector fields $\overline{\phi}^\mu$, η^μ and two scalar fields p, q. They have ghost numbers -1, 0; -2, -1; -2, -1, respectively. Among them, b^+, $\overline{\phi}$ and p are bosonic; the rest are fermionic. The transformation rules are

$$\delta \chi^{+}_{\alpha\beta\gamma\delta} = b^{+}_{\alpha\beta\gamma\delta}, \quad (2.12)$$

$$\begin{aligned}\delta b^{+}_{\alpha\beta\gamma\delta} &= \chi^{+}_{\alpha\beta\gamma\delta;\rho}\phi^\rho + \chi^{+}_{\rho\beta\gamma\delta}\phi^\rho{}_{;\alpha} \\ &\quad + \chi^{+}_{\alpha\rho\gamma\delta}\phi^\rho{}_{;\beta} + \chi^{+}_{\alpha\beta\rho\delta}\phi^\rho{}_{;\gamma} + \chi^{+}_{\alpha\beta\gamma\rho}\phi^\rho{}_{;\delta};\end{aligned} \quad (2.13)$$

$$\delta \overline{\phi}^\mu = \eta^\mu, \quad (2.14)$$

$$\delta \eta^\mu = \overline{\phi}^\mu{}_{;\nu}\phi^\nu - \phi^\mu{}_{;\nu}\overline{\phi}^\nu; \quad (2.15)$$

$$\delta p = q, \quad (2.16)$$

$$\delta q = \phi^\lambda p_{,\lambda}. \quad (2.17)$$

After a tedious but straightforward calculation, the total Lagrangian is

$$\begin{aligned}\mathcal{L} &= \sqrt{g}\, b'^{+\alpha\beta\gamma\delta} W^{+}_{\alpha\beta\gamma\delta} + \sqrt{g}\,\chi^{+\alpha\beta\gamma\delta}(D\psi)_{\alpha\beta\gamma\delta} + \sqrt{g}\,\eta'_\lambda (K^*\psi)^\lambda \\ &\quad + \sqrt{g}\,\overline{\phi}_\lambda (K^* K\phi)^\lambda - 2\sqrt{g}\,\overline{\phi}_\lambda [(\psi^\lambda{}_{\mu;\nu} - \frac{1}{2}\psi_{\mu\nu}{}^{;\lambda})\psi^{\mu\nu} + \psi^\mu{}_{\mu,\nu}\psi^{\lambda\nu}] \\ &\quad + \sqrt{g}\, q \psi^\lambda{}_\lambda,\end{aligned} \quad (2.18)$$

with the field redefinitions

$$b'^{+}_{\alpha\beta\gamma\delta} = b^{+}_{\alpha\beta\gamma\delta} + \tfrac{1}{2}\psi^\rho{}_\rho \chi^{+}_{\alpha\beta\gamma\delta} - 5\psi_\alpha{}^\rho \chi^{+}_{\rho\beta\gamma\delta}, \quad (2.19)$$

$$\eta'^\lambda = \eta^\lambda + \tfrac{1}{2}\psi^\rho{}_\rho \overline{\phi}^\lambda + \psi^\lambda{}_\rho \overline{\phi}^\rho. \quad (2.20)$$

In (2.23), D is a second order differential operator defined by

$$\begin{aligned}(D\psi)_{\alpha\beta\gamma\delta} &= -(P_{\alpha\beta\gamma;\delta} - P_{\alpha\beta\delta;\gamma}) \\ &\quad + \tfrac{1}{4}(R_{\alpha\gamma}\psi_{\beta\delta} - R_{\alpha\delta}\psi_{\beta\gamma} - R_{\beta\gamma}\psi_{\alpha\delta} + R_{\beta\delta}\psi_{\alpha\gamma}).\end{aligned} \quad (2.21)$$

$D\psi = 0$ is the linearization of the anti-self-dual equation $W^+ = 0$ at an anti-self-dual metric [7]. However, the metric $g_{\mu\nu}$ in the Lagrangian (2.18) is arbitrary.

Apart from the well-known constraints such as $W^+ = 0$, $D\psi = 0$, $K^*\psi = 0$ and $\psi^\lambda_\lambda = 0$, one of the classical equations of motion from (2.18) is

$$\phi^\lambda(x) = -2\int d^4y\,\sqrt{g}\,G^\lambda{}_\rho(x,y)\left[(\psi^\rho{}_{\mu;\nu} - \tfrac{1}{2}\psi_{\mu\nu}{}^{;\rho})\psi^{\mu\nu} + \psi^\mu{}_{\mu,\nu}\psi^{\rho\nu}\right](y), \tag{2.22}$$

where $G^\mu{}_\nu(x,y)$ is the Green's function of the Laplacian K^*K acting on vector fields. In the semiclassical limit, this is also the formula for the expectation value of ϕ^λ after integrating over the non-zero modes.

3. Curvature of the universal fibration

The universal bundle in gauge theory was first studied in the context of the family index theorem and the axial anomaly [8]. Recently, it was shown that the BRST symmetry in topological Yang–Mills theory can have the geometric interpretation of the connection and curvature of the universal bundle [6]. Gauge and gravitational anomalies and universal bundles were also studied using the evaluation map in field theory [9]. This approach was used to obtain the BRST algebra of topological Yang–Mills theory [10]. In gravitational problems, one often makes use of the frame bundle of the spacetime manifold and the corresponding classifying space [9]. In fact, the universal bundle for two dimensional gravity has been worked out using the orthonormal frame bundle [11]. In this section, we consider the universal fibration of Riemannian manifolds using the set of Riemannian metrics only.

Let M be a finite dimensional compact oriented smooth manifold, $\mathcal{M}et$ the set of Riemannian metrics on M, and $\mathcal{D}iff^+$ the group of diffeomorphisms of M which preserve the orientation. We assume that $\mathcal{D}iff^+$ acts on $\mathcal{M}et$ freely, i.e., no Killing fields exist for any Riemannian metric g on M. Otherwise, we should restrict either $\mathcal{M}et$ to the subset of metrics with no Killing fields or $\mathcal{D}iff^+$ to $\{f : M \to M \mid f(x_0) = x_0, df_{x_0} = \mathrm{id}_{T_{x_0}M}\}$ for a chosen point $x_0 \in M$. Then the quotient $\mathcal{M}et/\mathcal{D}iff^+$ is a smooth Hilbert manifold [12] and the group $\mathcal{D}iff^+$ also acts on $M \times \mathcal{M}et$ freely via $f \in \mathcal{D}iff^+ : (x,g) \mapsto (f^{-1}x, f^*g)$. Hence we have a smooth fibration

$$(M \times \mathcal{M}et)/\mathcal{D}iff^+ \xrightarrow{\pi} \mathcal{M}et/\mathcal{D}iff^+, \tag{3.1}$$

called the *universal fibration of Riemannian manifolds*.

There is a tautological Riemannian metric on each fiber defined as follows: Given an element $[g] \in \mathcal{M}et/\mathcal{D}iff^+$ in the base, we pick a representative $g \in [g]$. The map $\varphi : \pi^{-1}([g]) \to M$, $\varphi : [(x,g)] \mapsto x$ induces a Riemannian metric φ^*g on the fiber $\pi^{-1}([g])$. It is independent of the choice of the representative. To see this, pick another $g' = f^*g \in [g]$, where $f \in \mathcal{D}iff^+$. Then the new map $\varphi' : \pi^{-1}([g]) \to M$ is $\varphi' = f^{-1} \circ \varphi$. However, the induced metric $\varphi'^* g' = \varphi^*(f^{-1})^* f^* g = \varphi^* g$ has not been changed. So the tautological metric on each fiber is well defined.

Secondly, there is a natural choice of horizontal spaces of the fibration (3.1). For any metric $g \in \mathcal{M}et$, the tangent space $T_g\mathcal{M}et \cong \{S \mid S \in \mathrm{Sym}^2(T^*M)\}$. We define a horizontal slice $T_g^H\mathcal{M}et = \{S \mid K^*S = 0\} \subset T_g\mathcal{M}et$, where K is the Killing operator. The slice theorem [12] says that there is an open neighborhood $U \subset T_g^H\mathcal{M}et$ such that $U \subset \mathcal{M}et \xrightarrow{\pi} \overline{U} \subset \mathcal{M}et/\mathcal{D}iff^+$ is a diffeomorphism. Therefore the map φ defined earlier extends to $(M \times \mathcal{M}et)/\mathcal{D}iff^+|_{\overline{U}} \xrightarrow{\varphi} M \times U$, a coordinate chart on $(M \times \mathcal{M}et)/\mathcal{D}iff^+$. Its differentiation is an isomorphism of tangent spaces

$$T_{[(x,g)]}(M \times \mathcal{M}et)/\mathcal{D}iff^+ \xrightarrow{\mathrm{d}\varphi} T_xM \oplus T_g^H\mathcal{M}et.$$

The horizontal space of the fibration (3.1) at $[(x,g)]$ is defined as $(\mathrm{d}\varphi)^{-1} \times T_g^H\mathcal{M}et$. The following commutative diagram

$$\begin{array}{ccc} (M \times \mathcal{M}et)/\mathcal{D}iff^+|_{\overline{U}} & \xrightarrow{\varphi} & M \times U \\ & \varphi' \searrow & \downarrow f^{-1}\times f^* \\ & & M \times U' \end{array} \tag{3.2}$$

for any $f \in \mathcal{D}iff^+$ and its differentiation

$$\begin{array}{ccc} T_{[(x,g)]}(M \times \mathcal{M}et)/\mathcal{D}iff^+ & \xrightarrow{\mathrm{d}\varphi} & T_xM \oplus T_g^H\mathcal{M}et \\ & \mathrm{d}\varphi' \searrow & \downarrow \mathrm{d}f^{-1}\oplus \mathrm{d}f^* \\ & & T_{f^{-1}x}M \oplus T_{f^*g}^H\mathcal{M}et \end{array} \tag{3.3}$$

imply that the horizontal space is well defined.

Following Bismut [13], a fibration $E \xrightarrow{\pi} B$ with typical fiber M is called a *fibration of Riemannian manifolds* if there is a Riemannian metric on each fiber and a choice of horizontal spaces. Since we know how to parallel transport the vertical vectors both within a fiber (given the Riemannian metric) and away from it (given the choice of horizontal spaces), there is a canonical connection $\nabla^{E/B}$ on the relative tangent bundle $T_*(E/B)$ over E, a sub-bundle of T_*E, whose fibers are tangent spaces to the fibers of the original bundle. Alternatively, the connection can be defined as a projection of the Levi-Civita connection on E. Choose a Riemannian metric g_B on B. Then the metrics on the fiber and the horizontal spaces lead to a Riemannian metric on E. An equivalent definition is $\nabla^{E/B} \equiv P \cdot \nabla^E \cdot P$, where ∇^E is the Levi-Civita connection on E and P is the projection on the fiber direction [13]. It turns out that the right hand side is independent of the metric g_B on B (see ref. [14] for a review).

The fibration (3.1) is universal in the sense that for any fibration of Riemannian manifolds $\pi : E \to B$ there exists a bundle map $E \to (M \times \mathcal{M}et)/\mathcal{D}iff^+$

under which both the Riemannian metrics of the fibers and the horizontal spaces of E are pull-backs from the universal fibration. Therefore the connection and curvature of $T_*(E/B)$ are the pull-backs of the universal ones, which we now calculate.

Suppose we want to evaluate the curvature at a point in $\pi^{-1}([g_0])$. Choose a basis $\{\partial_\mu, S \,|\, \mu = 1, \ldots, \dim M, S \in T^H_{g_0}\mathcal{M}et\}$ for the tangent space $T_*(M \times \mathcal{M}et)/\mathcal{D}iff^+$. Any pair of vectors in this basis has a vanishing Lie bracket. Along the fiber direction, the Christoffel symbols $\tilde{\Gamma}^\mu_{\lambda\nu} = \Gamma^\mu_{\lambda\nu}$ and the curvature tensor

$$\tilde{R}^\mu{}_{\nu\rho\sigma} = R^\mu{}_{\nu\rho\sigma} \tag{3.4}$$

are standard. The covariant derivative along the horizontal direction are characterized by

$$\tilde{\Gamma}^\mu_{S\nu} = \tfrac{1}{2} g^{\mu\lambda}(g_{\lambda\nu,S} + \tilde{g}_{\lambda S,\nu} - \tilde{g}_{\nu S,\lambda}), \tag{3.5}$$

where the subscript $,S$ denotes the functional derivative with respect to the metric variation S. Since the connection we compute is independent of the metric on the base manifold, $\tilde{g}_{\mu S}$ can be chosen as the inner product of ∂_μ and the projection of S to the vertical direction. In the fiber $\pi^{-1}([g_0])$, $\tilde{g}_{\mu S} = 0$, so

$$\tilde{\Gamma}^\mu_{S\nu} = \tfrac{1}{2} g^{\mu\lambda} S_{\lambda\nu} = \tfrac{1}{2} S^\mu{}_\nu. \tag{3.6}$$

Using (3.22), the curvature two-form contracted with $S \in T^H_{g_0}\mathcal{M}et$ and $\partial_\lambda \in T_xM$ is

$$\begin{aligned} \tilde{R}^\mu{}_{\nu S\lambda} &= \Gamma^\mu_{\lambda\nu,S} - \tilde{\Gamma}^\mu_{S\nu,\lambda} + \tilde{\Gamma}^\mu_{S\rho}\Gamma^\rho_{\lambda\nu} - \Gamma^\mu_{\lambda\rho}\tilde{\Gamma}^\rho_{S\nu} \\ &= \tfrac{1}{2}(S_\lambda{}^\mu{}_{;\nu} - S_{\lambda\nu}{}^{;\mu}). \end{aligned} \tag{3.7}$$

At a point $[(x,g)]$ away from the fiber $\pi^{-1}([g_0])$, the vector $(0,S) \in T_xM \oplus T_g\mathcal{M}et$ is not necessarily horizontal. Instead, it has a decomposition

$$(0,S) = (GK^*S(x),0) + (0,S - KGK^*S) + (-GK^*S(x),KGK^*S). \tag{3.8}$$

The first and second terms are vertical and horizontal, respectively. The third one is tangent to the orbit of $\mathcal{D}iff^+$ and vanishes after taking the quotient. Therefore $\tilde{g}_{\lambda S} = (GK^*S)_\lambda$ and

$$\tilde{\Gamma}^\mu_{S\nu} = \tfrac{1}{2}[S^\mu{}_\nu + (GK^*S)^\mu{}_{;\nu} - (GK^*S)_\nu{}^{;\mu}]. \tag{3.9}$$

The universal curvature contracted with S, $T \in T^H_{g_0}\mathcal{M}et$ is

$$\begin{aligned} \tilde{R}^\mu{}_{\nu ST} &= \tilde{\Gamma}^\mu_{T\nu,S} - \tilde{\Gamma}^\mu_{S\nu,T} + \tilde{\Gamma}^\mu_{S\lambda}\tilde{\Gamma}^\lambda_{T\nu} - \tilde{\Gamma}^\mu_{T\lambda}\tilde{\Gamma}^\lambda_{T\mu} \\ &= \tfrac{1}{2}[-S^{\mu\lambda}T_{\lambda\nu} + (GK^*T_{,S})^\mu{}_{;\nu} - (GK^*T_{,S})_\nu{}^{;\mu}] + \tfrac{1}{4}S^\mu{}_\lambda T^\lambda{}_\nu - (S \leftrightarrow T) \\ &= -\tfrac{1}{4}(S^{\mu\lambda}T_{\lambda\nu} - T^{\mu\lambda}S_{\lambda\nu}) + (Gw(S,T))^\mu{}_{;\nu} - (Gw(S,T))_\nu{}^{;\mu}, \end{aligned} \tag{3.10}$$

where $G = (K^*K)^{-1}$ acts on the vector $w(S,T)$ defined by

$$w(S,T)^\lambda = [(S^\lambda{}_{\mu;\nu} - \tfrac{1}{2}S_{\mu\nu}{}^{;\lambda})T^{\mu\nu} + S^\mu{}_{\mu,\nu}T^{\lambda\nu}] - (S \leftrightarrow T). \tag{3.11}$$

The components of the universal curvature (3.7), (3.10), (3.11) reproduce the physical calculations (2.7), (2.8), (2.22). The zero modes of $\psi_{\mu\nu}$ tangent to the moduli space is $S_{\mu\nu}$. The BRST invariant bosonic field ϕ^λ of ghost number 2 corresponds to $-Gw(S,T)^\lambda$.

If we would consider the universal bundle of orthonormal frames [11], the first term in (3.10) would be absent. This discrepancy might be a reflection of the ordering ambiguity familiar to physicists. Although Ashtekar's variables and the vierbein formalism describe the same classical theory of gravity as the standard tensor notation does, the operator ordering could be completely different upon quantization. Mathematically, this corresponds to the existence of more than one natural bundle over the same moduli space (see ref. [9], for example, for the construction of various bundles). It should also be pointed out that after an earlier version of this paper was written, another study within the context of the universal bundle, but from a different perspective has appeared [15].

4. Moduli space, conformal gauge fixing and observables

Four dimensional topological gravity is one of the theories in which the stationary phase approximation is exact. For the bosonic fields, the configuration of classical extrema is the moduli space of anti-self-dual conformal structures [7,16]

$$\mathcal{M} = \{g \mid W^+ = 0\}/(\mathcal{D}iff^+ \times \mathcal{C}onf). \tag{4.1}$$

Early examples of four-manifolds with non-empty moduli space $\mathcal{M}$ are $\overline{\mathbb{CP}^2}$, K3 in addition to the conformally flat ones like $\mathbb{R}^4$, S^4, T^4 and $S^1 \times S^3$. The list was extended to the connected sums $\#2\overline{\mathbb{CP}^2}$, $\#3\overline{\mathbb{CP}^2}$ [17], $\#n\overline{\mathbb{CP}^2}$ [18] and $N\mathrm{K3}\#n\overline{\mathbb{CP}^2}$ ($n \geq 2N+1$) [19]. Recently, Taubes has shown that for any four-manifold M, anti-self-dual metrics exist on $M\#n\overline{\mathbb{CP}^2}$ for sufficiently large n [20].

The fermionic zero modes satisfying the constraint equations live in the cohomology of the elliptic complex

$$0 \to \mathrm{Vect}(M) \xrightarrow{K_0} \mathrm{Sym}_0^2(T^*M) \xrightarrow{D} \mathrm{Sym}_0^2(\Lambda_+^2(T^*M)) \to 0, \tag{4.2}$$

where Sym_0^2 denotes the traceless part of the symmetric product Sym^2, Λ_+^2 is the self-dual part of the anti-symmetric product, and K_0 is the projection of the Killing operator to the traceless part, i.e., the conformal Killing operator. Note that $K^*\psi$ and $K_0^*\psi$ coincide if ψ is traceless. The complex (4.3) has two alternative forms:

$$0 \to \mathrm{Vect}(M) \oplus \Gamma(\mathbb{R}) \xrightarrow{K\oplus i} \mathrm{Sym}^2(T^*M) \xrightarrow{D} \mathrm{Sym}_0^2(\Lambda_+^2(T^*M)) \to 0, \tag{4.3}$$

where i is the inclusion of $\Gamma(\mathbb{R})$ into the trace part of $\mathrm{Sym}^2(T^*M)$, and

$$0 \to \mathrm{Vect}(M) \xrightarrow{K} \mathrm{Sym}^2(T^*M) \xrightarrow{D\oplus\delta R} \mathrm{Sym}_0^2(\Lambda_+^2(T^*M)) \oplus \Gamma(\mathbb{R}) \to 0, \tag{4.4}$$

where δR is the linearization of the scalar curvature. The index of all these elliptic complexes is [21] (see also refs. [22,18,16])

$$\text{Index} = \tfrac{1}{2}[29\tau(M) + 15\chi(M)]. \tag{4.5}$$

In the present context, the signature and the Euler characteristics of a four-manifold M can be written as (see, for example, refs. [7,16])

$$\tau(M) = \frac{1}{12\pi^2}\int_M \mathrm{d}^4x\,\sqrt{g}\,(|W^+|^2 - |W^-|^2), \tag{4.6}$$

$$\chi(M) = \frac{1}{8\pi^2}\int_M \mathrm{d}^4x\,\sqrt{g}\,(|W^+|^2 + |W^-|^2 - |\mathrm{Ric}_0|^2 + \tfrac{1}{24}R^2), \tag{4.7}$$

where Ric_0 is the traceless part of the Ricci tensor.

Further assumptions should be made to avoid singularities of $\mathcal{M}$. First, we assume that K_0 has no zero modes, that is, M should have no conformal Killing vectors for any metric. This is the analog of having no reducible connections in gauge theory. Unfortunately, the assumption is not valid for many four-manifolds such as S^4, T^4 and $\overline{\mathbb{C}P^2}$ [18]. Though a slightly milder assumption exists (namely, the conformal Killing group has a constant dimension for all metrics on M), we are not going to pursue the consequences here. Secondly, we assume that D is onto. While an analogous condition in gauge theory holds for generic metrics, the assumption here is false at least for T^4 and K3 [18]. (The case T^4 has been considered in ref. [4].) Assuming that neither these subtleties occur, the moduli space $\mathcal{M}$ is a smooth manifold of dimension

$$\dim\mathcal{M} = -\tfrac{1}{2}(29\tau + 15\chi). \tag{4.8}$$

Its tangent space is spanned by the zero modes of the fermion fields ψ.

From the physics point of view, the three different forms of the fundamental elliptic complex (4.2)–(4.4) correspond to three different treatments of the conformal symmetry, which we comment in the same order below. That they have the same index confirms that the physics is independent of the gauge fixing. First, we can write the BRST transformation (2.1) and add a traceless constraint $\psi^\lambda_\lambda = 0$ as in some of the earlier attempts or in section 2. Alternatively, conformal and diffeomorphism symmetries can be treated on the same footing, i.e., by $g_{\mu\nu} = \psi_{\mu\nu} + c_{\mu;\nu} + c_{\nu;\mu} + c' g_{\mu\nu}$, supplemented by the BRST transformations of other fields. And finally, conformal gauge fixing can be achieved by the constraint $R + \alpha = 0$ (α is a constant) [4], due to Schoen's solution to the Yamabe problem: any Riemannian metric can be conformally deformed to one with constant scalar curvature [23]. Indeed, if the constant scalar curvature is non-positive, then two conformally equivalent metrics of the same volume must be the same. However, more careful analysis reveals that, if there is a metric with positive constant scalar curvature, then the manifold (if simply connected) has to be the connected sum of $\overline{\mathbb{C}P^2}$ [7], and there seems to be no understanding about the uniqueness. Therefore, of the three possibilities $R = -1, 0, 1$, $R = 0$ has to be supplemented by the normalization of the volume [4], whereas for $R = 1$, the extra redundancy is not clear.

To study the intersecting theory on the moduli space, we consider the integration of differential forms along the fiber of (3.1). Since the orientation on M induces one on each fiber, any differential n-form on the total space $(M \times \mathcal{M}et)/\mathcal{D}iff^+$ yields an $(n-4)$-form on the base $\mathcal{M}et/\mathcal{D}iff^+$. However, for lower dimensional cycles ω in M, two problems occur. First, there is no canonical embedding of $\omega \subset M$ in the fiber; this makes the integration ambiguous. Secondly, the bi-grading of differential forms on $U \times M$ depends on the particular coordinate chart we choose. Therefore the dependence of the observables on the diffeomorphism ghost c^λ seems necessary. On one hand, the ghost compensates the variation of the lower dimensional cycle ω under diffeomorphism transformations. On the other hand, the ghost field c^λ mixes the grading of differential forms on $U \times M$ in such a way that the variation of the observables under an infinitesimal diffeomorphism is BRST trivial [4], i.e., for any vector field v, $\int_\omega L_v W_k = \int_\omega i_v \mathrm{d}W_k = -\delta \int_\omega i_v W_{k+1}$. Therefore, if ω is a k-cycle in M, the expectation value $\langle \mathcal{O}(\omega) \rangle$ of $\mathcal{O}(\omega) = \int_\omega W_k$ depends only on the homology class $[\omega] \in H_k(M)$.

One should also require that $\langle \mathcal{O}(\omega) \rangle$ is invariant under large diffeomorphisms to avoid global gravitational anomalies. The invariant set $H_*(M)^{\mathcal{D}iff^+}$ fixed by the orientation preserving diffeomorphism group $\mathcal{D}iff^+$ is usually small. To construct more general $\mathcal{D}iff^+$-invariant observables, we consider the graded polynomial algebra $\hat{\mathbb{R}}[H_*(M)]$ over the homology group $H_*(M)$ with real coefficients. A typical graded monomial is $\omega_1 \hat{\otimes} \cdots \hat{\otimes} \omega_r$, where the ω_i's are homology k_i-cycles satisfying the graded commutativity $\omega_i \hat{\otimes} \omega_j = (-1)^{k_i k_j} \omega_j \hat{\otimes} \omega_i$ and associativity. To any $\mathcal{D}iff^+$-invariant graded polynomial $P = \omega_1 \hat{\otimes} \cdots \hat{\otimes} \omega_r + \cdots$, we associate an observable

$$\mathcal{O}(P) = \prod_{i=1}^{r} \int_{\omega_i} W_{k_i} + \cdots . \tag{4.9}$$

It is clearly invariant under $\mathcal{D}iff^+$. For simply connected compact four-manifolds, the only interesting part of $\hat{\mathbb{R}}[H_*(M)]$ is $\mathbb{R}[H_2(M)]$, the polynomial algebra over $H_2(M)$. The invariant part $\mathbb{R}[H_2(M)]^{\mathcal{D}iff^+}$ contains $\mathbb{R}[q_M]$, where q_M is the intersecting form of M. For a large class of algebraic surfaces, $\mathbb{R}[H_2(M)]^{\mathcal{D}iff^+}$ is either $\mathbb{R}[q_M, k^2]$ or $\mathbb{R}[q_M, k]$, where k is the first Chern class of the canonical bundle [24], though little is known about $\hat{\mathbb{R}}[H_*(M)]^{\mathcal{D}iff^+}$ in the general case.

The author would like to thank R. Friedman, N. Leung, K. Li, R. Myers and

I. M. Singer for helpful discussions, and the referee for bringing to his attention ref. [9]. The work is supported in part by NSF grant DMS-8505550 through a MSRI Postdoctoral Fellowship.

References

[1] E. Witten, Topological quantum field theory, Commun. Math. Phys. 117 (1988) 353–386.
[2] E. Witten, Topological gravity, Phys. Lett. B 206 (1988) 601–606.
[3] J.M.F. Labastida and M. Pernici, A Lagrangian for topological gravity and its BRST quantization, Phys. Lett. B 213 (1988) 319–324;
R. Brooks, D. Montano and J. Sonnenschein, Gauge fixing and renormalization in topological quantum field theory, Phys. Lett. B 214 (1988) 91–97.
[4] R. Myers and V. Periwal, Topological gravity and moduli space, Nucl. Phys. B 333 (1990) 536–550; Invariants of smooth 4-manifolds from topological gravity, Nucl. Phys. B 361 (1991) 290–310;
R. Myers, On alternate formulations of topological gravity, Phys. Lett. B 252 (1990) 365–369; New observables for topological theories, Nucl. Phys. B 343 (1990) 705–715.
[5] C.G. Torre, A topological field theory of gravitational instantons, Phys. Lett. B 252 (1990) 242–246;
H. Kunitomo, Topological half-flat gravity, Mod. Phys. Lett. A 6 (1990) 2389–2396.
[6] L. Baulieu and I.M. Singer, Topological Yang–Mills theory, in: *Conformal field theories and related topics*, eds. P. Binétruy et al., Nucl. Phys. B (Proc. Suppl.) 5B (North-Holland, Amsterdam, 1988) pp. 12–19.
[7] M. Itoh, Moduli space of half conformally flat structures, Max-Plank-Institut für Mathematik preprint MPI/90-56 (1990).
[8] M.F. Atiyah and I.M. Singer, Dirac operator coupled to vector potentials, Proc. Natl. Acad. Sci. USA 81 (1984) 2597–2600.
[9] L. Bonora and P. Cotta-Ramusino and M. Rinaldi and J. Stasheff, The evaluation map in field theory, sigma-models and strings I, Commun. Math. Phys. 112 (1987) 237–282; II, Commun. Math. Phys. 114 (1988) 381–437.
[10] H. Kanno, Observables in topological Yang–Mills theory and the Gribov problem, Lett. Math. Phys. 19 (1990) 249–255.
[11] L. Baulieu and I.M. Singer, Conformally invariant gauge fixed actions for 2-D topological gravity, Commun. Math. Phys. 135 (1991) 253–265.
[12] D.G. Ebin, On the space of Riemannian metrics, Bull. Amer. Math. Soc. 74 (1968) 1001–1003; The manifold of Riemannian metrics, in: *Global Analysis*, eds. S.S. Chern and S. Smale, Proc. Symp. Pure Math. 15 (Amer. Math. Soc., Providence, 1970) pp. 11–40;
A.E. Fischer, The theory of superspace, in: *Relativity*, eds. M. Carmeli, S.I. Fickler and L. Witten, Proc. Relativity Conf. in the Midwest (Plenum, New York, 1970) pp. 303–357.
[13] J.-M. Bismut, The index theorem for families of Dirac operators: two heat equation proofs, Invent. Math. 83 (1986) 91–151.
[14] N. Berline, E. Getzler and M. Vergne, *Heat kernels and Dirac operators* (Springer, Berlin, 1992) ch. 10.
[15] D. Birmingham and M. Rakowski, A field theoretic realization of a universal bundle for gravity, Phys. Lett. B 275 (1991) 289–294.
[16] A.D. King and D. Kotschick, The deformation theory of anti-self-dual conformal structures, Math. Ann. 294 (1992) 591–609.
[17] Y.S. Poon, Compact self-dual manifolds with positive scalar curvature, J. Diff. Geom. 24 (1986) 97–132; Small resolutions of double solids as twistor spaces, Rice University mathematics preprint (1987).
[18] A. Floer, Self-dual conformal structures on $l\mathbb{C}P^2$, J. Diff. Geom. 33 (1991) 551–573.
[19] S. Donaldson and R. Friedman, Connected sums of self-dual manifolds and deformations of singular spaces, Nonlinearity 2 (1989) 197–239.

[20] C.H. Taubes, The existence of anti-self-dual conformal structures, J. Diff. Geom. 36 (1992) 163–253.
[21] I.M. Singer, Lecture notes (unpublished).
[22] T. Eguchi, P.B. Gilkey and A.J. Hanson, Gravitation, gauge theory and differential geometry, Phys. Rep. 66 (1980) 213–393.
[23] H. Yamabe, On the deformation of Riemannian structures on compact manifolds, Osaka Math. J. 12 (1960) 21–37;
R. Schoen, Conformal deformation of a Riemannian metric to constant scalar curvature, J. Diff. Geom. 20 (1984) 479–495.
[24] R. Friedman and J. Morgan, Smooth four-manifolds and complex surfaces (in preparation), ch. 4.